View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

JOURNAL OF ALGEBRA **146**, 412–426 (1992)

A Pushing-Up Approach to the Quasithin Simple Finite Groups with Solvable 2-Local Subgroups

GOMI KENSAKU

Department of Mathematics, College of Arts and Sciences, University of Tokyo, Komaba, Meguro, Tokyo 153

AND

HAYASHI MAKOTO

Department of Mathematics, Aichi University of Education, Hirosawa, Igaya, Kariya 448

Communicated by Walter Feit

Received February 21, 1990

DEDICATED TO PROFESSOR TSUZUKU TOSIRO ON HIS SIXTIETH BIRTHDAY

1. INTRODUCTION

A finite group G of even order is said to be quasithin if $e(G) \leqq 2$, while G is thin if $e(G) \leqq 1$, where $e(G)$ is the maximum of the 2-local p-rank $m_{2,p}(G)$ as p ranges over all odd primes, and $m_{2,p}(G)$ is the maximum of the p-rank $m_p(L)$ as L ranges over all 2-local subgroups of G. The group G is said to be of characteristic 2 type if each 2-local subgroup L of G satisfies the condition $C_L(O_2(L)) \leqq O_2(L)$ or equivalently $F^*(L) = O_2(L)$.

The purpose of this paper is to give an alternative proof of the following theorem established by the combined work of Janko [1] and Smith [2].

THEOREM. *Let G be a quasithin nonabelian simple finite group of characteristic 2 type in which each 2-local subgroup is solvable. Then G is isomorphic to one of the groups A_6, $L_2(p)$, p a Fermat or Mersenne prime $\geqq 5$, $L_3(3)$, M_{11}, $U_3(3)$, $L_2(2^n)$, $Sz(2^{2n-1})$, $U_3(2^n)$, $n \geqq 2$, and ${}^2F_4(2)'$.*

This theorem forms the starting point for Mason's analysis to determine all quasithin simple finite groups of characteristic 2 type. Mason's result (unpublished) is one of the major steps in the classification of the simple finite groups. For an outline of the classification, the reader may consult

0021-8693/92 \$3.00
Copyright © 1992 by Academic Press, Inc.
All rights of reproduction in any form reserved.

the Proceedings [3, 4] of the Durham Conference and the Santa Cruz Conference on finite groups.

In a previous paper [5], we have given an alternative proof of the theorem of Janko by making use of a pushing-up theorem. Our aim is to pursue the same method to obtain a unified proof of the theorems of Janko and Smith. Thus our method is keeping aloof from Thompson's method [6] which Janko and Smith have used.

The invariant $f(G)$ defined as follows is more pertinent to our method than $e(G)$. If G is a finite group and p is a prime, we shall denote by $n_p(G)$ the maximum rank of a chief p-factor of G. When G has even order, we define $n_{2,p}(G)$ to be the maximum of $n_p(L)$ as L ranges over all 2-local subgroups of G, and then define $f(G)$ to be the maximum of $n_{2,p}(G)$ as p ranges over all odd primes. Thompson [6] observed that if $e(G) \leqq 2$ then $f(G) \leqq 2$. Our method is easily extended to the case $f(G) \leqq 2$.

Our notation is standard except the following: If S is a subgroup of a group G, $\langle S^G \rangle$ denotes the subgroup generated by all G-conjugates of S and $\bigcap S^G$ denotes the intersection of all G-conjugates of S. We shall not use G' for the commutator subgroup $[G, G]$.

2. Pushing-Up Theorems

If G is a group and S is its subgroup, we shall denote by $O_S(G)$ the unique maximal normal subgroup of G contained in S. A pushing-up theorem is aimed at answering the following fundamental question about $O_S(G)$.

Question. *Let G be a group and let X_i, $i = 1, 2, \ldots, n$, be finite subgroups of G which have a common subgroup S such that $O_S(X_i) \neq 1$ for each i and $O_S(\langle X_1, \ldots, X_n \rangle) = 1$. Determine the structure of the X_i, $i = 1, 2, \ldots, n$.*

The pushing-up theorems which are used in this paper are all concerned with the following definitions.

Definition. Let X be a finite group and S be a subgroup. The group X is S-irreducible if S is contained in a unique maximal subgroup of X. The group X is 2-irreducible if X is S-irreducible for $S \in \mathrm{Syl}_2(X)$.

Definition. If finite subgroups X_1, X_2, and S of some group G satisfy the condition $S \in \mathrm{Syl}_2(X_i)$, $i = 1, 2$, we shall call the triple (X_1, S, X_2) a Sylow 2-amalgam or an S_2-amalgam in G. If furthermore X_i is S-irreducible and $O_S(X_i) \neq 1$ for each i while $O_S(\langle X_1, X_2 \rangle) = 1$, we shall say that the S_2-amalgam is extremal.

As in [5], we shall limit our proof of the Janko–Smith theorem to the determination of the 2-local structure of the group G under investigation to the extent necessary to identify G by some characterization theorems. The starting point of this 2-local analysis is the following theorem [7].

THEOREM A. *Let G be a group of characteristic* 2 *type and let $S \in \mathrm{Syl}_2(G)$. If each 2-local subgroup of G is solvable, then one of the following holds*:

(A.1) *There is an extremal S_2-amalgam (X_1, S, X_2) in G.*

(A.2) *S is contained in a unique maximal 2-local subgroup of G.*

The goal of our 2-local analysis is the following two theorems.

THEOREM B. *In case* (A.1) *of Theorem* A, *if in addition $f(G) \leqq 2$, then the X_i, $i = 1, 2$, are thin.*

THEOREM C. *In case* (A.2) *of Theorem* A, *one of the following holds*:

(C.1) *The unique maximal 2-local subgroup of G containing S is a strongly embedded subgroup, which is proper if $O_2(G) = 1$.*

(C.2) *S is dihedral or semidihedral or isomorphic to a Sylow 2-subgroup of* $\mathrm{Aut}(Sp_4(2))$.

It is evident that Theorem C gives enough information about case (A.2): if G is simple, we can identify G by appealing to well-known characterization theorems, with the help of transfer lemmas when S is of type $\mathrm{Aut}(Sp_4(2))$. Also Theorem B gives almost enough information about case (A.1) with $f(G) \leqq 2$, for the possible structure of the groups X_i and S in Theorem B is determined completely by the following Theorem D [8–10] and hence if G is simple, we shall be able to identify G by some extra work and appealing to some characterization theorems. Theorem D will also be used in the proofs of Theorems B and C.

THEOREM D. *Let (X_1, S, X_2) be an extremal S_2-amalgam in some group satisfying the following conditions*:

(D.1) $C_{X_i}(O_2(X_i)) \leqq O_2(X_i)$, $i = 1, 2$.

(D.2) *X_i is solvable and thin, $i = 1, 2$.*

Then (X_1, S, X_2) or (X_2, S, X_1) is isomorphic to one of the $GL_3(2)$-amalgam, $Sp_4(2)$-amalgam, $G_2(2)'$-amalgam, $G_2(2)$-amalgam, M_{12}-amalgam, $\mathrm{Aut}(M_{12})$-*amalgam, ${}^2F_4(2)'$-amalgam, and ${}^2F_4(2)$-amalgam.*

The reader may consult [9, II] for the definitions not given above.

Having shown that Theorems B and C serve our purpose, we conclude with two more pushing-up theorems of Tanaka [11] and Gomi [12].

THEOREM E. *Let* (X_1, S, X_2) *be an extremal* S_2*-amalgam in some group satisfying* (D.1) *and the following*:

(E.1) X_i *is solvable and* $f(X_i) \leqq 2$, $i = 1, 2$.

(E.2) $S = [O_2(X_1), O^2(X_1)](O_2(X_1) \cap O_2(X_2))[O_2(X_2), O^2(X_2)]$.

Then the X_i, $i = 1, 2$, *are thin, and hence the* S_2*-amalgam* (X_1, S, X_2) *is determined by Theorem* D.

We note that if an extremal S_2-amalgam (X_1, S, X_2) satisfies (D.1) and (D.2), then it satisfies (E.2) (and (E.1)). To prove this, observe that $[O_2(X_i), O^2(X_i)](O_2(X_1) \cap O_2(X_2))$ is normal in X_i and then use [9, I, 3.2.4] to show that it is equal to $O_2(X_i)$, $i = 1, 2$. As $S = O_2(X_1)\, O_2(X_2)$ by [9, I, 3.5], it follows that (E.2) is satisfied.

THEOREM F. *Let* G *be a solvable* 2*-irreducible group with* $C_G(O_2(G)) \leqq O_2(G)$, *and let* $S \in \mathrm{Syl}_2(G)$. *If no nonidentity characteristic subgroup of* S *is normal in* G, *then* $G = SJ(G)$ *and* $J(G)$ *is the direct product of one or more* (*isomorphic*) *copies of* S_4, *copies of* D_8, *and copies of* Z_2.

Here $J(G)$ is the Thompson subgroup generated by all elementary abelian 2-subgroups of G of maximal order.

3. SOLVABLE 2-IRREDUCIBLE GROUPS

In this section, G is a solvable 2-irreducible group, $S \in \mathrm{Syl}_2(G)$, and H is the unique maximal subgroup of G containing S.

3.1. (1) If $N \lhd G$, then either $S \cap N \lhd G$ or $O^2(G) \leqq N$.

(2) If $O_2(G) \leqq N \lhd G$ and $O^2(G) \nleqq N$, then $O_2(G) \in \mathrm{Syl}_2(N)$ and $O_2(G/N) = 1$.

(3) $\pi(G) \subset \{2, p\}$ for some odd prime p.

(4) If $S \lhd G$, then G/S is a cyclic p-group, $p \neq 2$.

Proof. (1) A Frattini argument gives $G = N_G(S \cap N)SN$ and so either $N_G(S \cap N) = G$ or $SN = G$ by the S-irreducibility of G. This shows that either $S \cap N \lhd G$ or $O^2(G) \leqq N$.

(2) As $O_2(G) \leqq S \cap N \lhd G$ by (1), $O_2(G) = S \cap N \in \mathrm{Syl}_2(N)$. Let $M/N = O_2(G/N)$. Then $O^2(G) \nleqq M$ an so $O_2(G) \in \mathrm{Syl}_2(M)$ by what we have shown. Hence, $M = O_2(G)N = N$.

(3) By Hall's theorem, S is contained in a Hall $\{2, p\}$-subgroup G_p of G for each odd prime p, and G is generated by all the G_p. Thus $G = G_p$ for some p by the S-irreducibility of G.

(4) If $S \lhd G$, then G/S is a 2-irreducible p-group, $p \neq 2$, so G/S has precisely one maximal subgroup and hence is cyclic.

3.2. If $\pi(G) \subset \{2, p\}$ and $P \in \mathrm{Syl}_p(G)$, then the following hold.

(1) $PO_2(G)/\Phi(P)\, O_2(G)$ is a chief factor of G with

$$C_G(PO_2(G)/\Phi(P)\, O_2(G)) = PO_2(G).$$

(2) $H = \Phi(P) S$.

(3) $O^2(G) = P[O_2(G), O^2(G)]$.

(4) If $N \lhd G$ and $O^2(G) \nleqq N$, then $(PN/N)/\Phi(PN/N) \cong P/\Phi(P)$.

Proof. (1) We may assume $O_2(G) = 1$. By 3.1.2 applied to $N = O_p(G)$, $G = O_{p,2}(G)$ and so $P = O_p(G) \lhd G$. Hence, $\Phi(P) \lhd G$ and $C_G(P/\Phi(P)) = P$. Suppose $L \lhd G$ and $\Phi(P) < L < P$. By Maschke's theorem, G has a normal subgroup M such that $P/\Phi(P) = L/\Phi(P) \times M/\Phi(P)$. As $G = PS$ and $P \cap S = 1$, LS and MS are proper subgroups of G generating G. This contradicts the S-irreducibility of G.

(2) We may assume $O_2(G) = 1$. We have $H = (P \cap H)S$ and $P \cap H \neq P$. Hence $P \neq (P \cap H)\, \Phi(P) \lhd \langle P, S \rangle = G$, so $P \cap H \leqq \Phi(P)$ by (1). Thus $H \leqq \Phi(P)S$. Conversely $\Phi(P)S$ is a proper subgroup of G by (1) and so $\Phi(P)S \leqq H$.

(3) This is because $P[O_2(G), O^2(G)] \lhd PO_2(G) \lhd G$.

(4) If $P \cap N \nleqq \Phi(P)$, then $N \nleqq \Phi(P)S = H$ and so $G = SN$, a contradiction. Thus $P \cap N \leqq \Phi(P)$, so $\Phi(P/P \cap N) = \Phi(P)/P \cap N$ and $(P/P \cap N)/\Phi(P/P \cap N) \cong P/\Phi(P)$.

3.3. Let $\pi(G) \subset \{2, p\}$ and assume $O_2(G) = 1$. Then $Z(S)$ is cyclic, and if $S \neq 1$, then the involution of $Z(S)$ inverts $O_p(G)/\Phi(O_p(G))$.

Proof. Regard $V = O_p(G)/\Phi(O_p(G))$ as a $GF(p)$-space. Then by 3.2.1, S is faithfully and irreducibly represented on V. By Schur's lemma, $Z(S)$ is cyclic, and if $F = GF(p)(\zeta)$, where ζ is a primitive $|Z(S)|$th root of unity in an algebraic closure of $GF(p)$, then V may be regarded as an FS-module so that for some generator z of $Z(S)$, $zv = \zeta v$ for all $v \in V$. Consequently the involution of $Z(S)$ inverts V.

3.4. If $zO_2(G)$ is an involution of $Z(S/O_2(G))$, then the following four conditions for $g \in G$ are equivalent.

(1) $g \notin H$.

(2) $G = \langle S, z^g \rangle$.

(3) $G = \langle S, S^g \rangle$.

(4) $G = \langle S, g \rangle$.

Proof. It suffices to prove that (1) implies (2), and we may assume $O_2(G) = 1$. Let $\pi(G) = \{2, p\}$ and $P \in \mathrm{Syl}_p(G)$. Write $g = sa$ with $s \in S$ and $a \in P$. Assume $g \notin H$. Then $a \notin \Phi(P)$ as $H = S\Phi(P)$ by 3.2.2. As $P = O_p(G)$ by 3.2.1, $a^z \equiv a^{-1} \bmod \Phi(P)$ by Section 3.3, so $zz^g = zz^a = a^{-z}a \equiv a^2 \bmod \Phi(P)$. If $z^g \in H$, then $a^2 \in H \cap P = \Phi(P)$, so $a \in \Phi(P)$, which is a contradiction. Thus $z^g \notin H$ and $G = \langle S, z^g \rangle$.

3.5. If $S/O_2(G)$ is a nonidentity elementary abelian group, then $G/O_2(G)$ is a dihedral group.

Proof. As $|S/O_2(G)| = 2$ by Section 3.3 applied to $G/O_2(G)$, 3.4.3 shows that $G/O_2(G)$ is generated by two involutions and hence dihedral.

3.6. If $S \neq O_2(G)$ and G has a noncentral chief 2-factor V with $|V| \leqq 8$, then $G/O_2(G)$ is a dihedral $\{2, 3\}$-group and $|V| = 4$.

Proof. Let $\pi(G) = \{2, p\}$ and $P \in \mathrm{Syl}_p(G)$. Let $*$ denote the natural homomorphism $G \to G/C_G(V)$. Then G^* is an irreducible subgroup of $GL(V)$, where V is regarded as a $GF(2)$-space. Consequently $O_2(G^*) = 1$, so $O_p(G^*) \neq 1$. Further since $O^2(G) \nleqq C_G(V)$ and $S \neq O_2(G)$, we have $S^* \neq 1$ by 3.1.2. The p-local structure of $GL_3(2) \cong L_2(7)$ now shows $G^* \cong D_6$, so $|V| = 4$ by the irreducibility of G^*. Further $P^* \cong Z_3$ and so P is a cyclic 3-group by 3.2.4. Hence, $|S/O_2(G)| = 2$ by 3.2.1 and thus $G/O_2(G)$ is dihedral by Section 3.5.

4. Pushing-Up and Pulling-Down

In this section, we shall study the following situation.

Hypothesis. *The finite group G is of characteristic 2 type and each 2-local subgroup of G is solvable.*

All propositions and definitions in this section will be stated under this hypothesis. They are designed to be used in possible future work.

Definitions. (1) If T is a 2-subgroup of G, we define $\Gamma(T)$ to be the set of all T-irreducible subgroups X of G with $T \in \mathrm{Syl}_2(X)$ and $C_X(O_2(X)) \leqq O_2(X)$.

(2) We shall denote by Ω the set of all ordered triples (X, T, Y) of a

2-subgroup T of G and a pair X, Y of elements of $\Gamma(T)$. We shall often write $X_T Y$ for (X, T, Y). If $\alpha = X_T Y \in \Omega$, we define $O_2(\alpha) = O_T(\langle X, Y\rangle)$.

(3) If subgroups X and Y of G satisfy the condition $[O_2(X), O^2(X)] \leqq O_2(Y)$, we shall say that X is 2-embedded into Y and write $X \leqq_2 Y$.

Remark. If $\alpha = (X, T, Y)$ is an extremal S_2-amalgam with $C_X(O_2(X)) \leqq O_2(X)$ and $C_Y(O_2(Y)) \leqq O_2(Y)$, then $\alpha \in \Omega$ and $O_2(\alpha) = 1$. The converse is also true.

4.1. If $\alpha = X_T Y \in \Omega$ and $O_2(\alpha) \neq 1$, then the following hold:

(1) $\langle X, Y\rangle$ is solvable.

(2) If $M = \langle O^2(Y)^{\langle X, Y\rangle}\rangle$ and M/N is a chief factor of $\langle X, Y\rangle$, then M/N is an elementary abelian q-group, where $\pi(Y) = \{2, q\}$.

(3) Either $X \leqq_2 Y$ or $Y \leqq_2 X$.

(4) $Z(T) \leqq O_2(\alpha)$.

Proof. As $O_2(\alpha) \neq 1$, $\langle X, Y\rangle$ is solvable and so, as $O^2(Y) \nleqq N$, M/N is an elementary abelian q-group. Set $L = \langle O^2(X)^{\langle X, Y\rangle}\rangle$. In proving (3) and (4), we may assume without loss that L is not a proper subgroup of M. Then $L \nleqq N$ and $M \nleqq N$, so $T \cap N \leqq O_2(\alpha)$ by 3.1.1. Hence, $[O_2(Y), O^2(Y)] \leqq T \cap M = T \cap N \leqq O_2(\alpha) \leqq O_2(X)$. Thus $Y \leqq_2 X$ and, as $C_Y(O_2(Y)) \leqq O_2(Y)$, $O^2(Y)$ is not contained in $C = C_{\langle X, Y\rangle}(O_2(\alpha))$. Hence, $T \cap C \lhd Y$ by 3.1.1. If $O^2(X) \nleqq C$, then also $T \cap C \lhd X$, so $T \cap C \leqq O_2(\alpha)$ and in particular $Z(T) \leqq O_2(\alpha)$. Therefore assume $O^2(X) \leqq C$. Then $[O_2(X), O^2(X)] \leqq T \cap C \leqq O_2(Y)$ and so $X \leqq_2 Y$ as well as $Y \leqq_2 X$. This yields that $O_2(X) \cap O_2(Y)$ is normal in $\langle X, Y\rangle$ and so equal to $O_2(\alpha)$. As $C_X(O_2(X)) \leqq O_2(X)$ and $C_Y(O_2(Y)) \leqq O_2(Y)$, $Z(T) \leqq O_2(\alpha)$ in this case as well.

4.2. Let $\alpha = X_T Y \in \Omega$ and assume that the following two conditions are satisfied:

(i) $O_2(\alpha) \neq 1$.

(ii) X is not 2-embedded into Y.

Set $M = \langle O^2(Y)^{\langle X, Y\rangle}\rangle$ and let M/N be a chief factor of $\langle X, Y\rangle$. Then the following hold:

(1) $\langle X, Y\rangle = XM$.

(2) $C_{O_2(X)}(M/N) = O_2(\alpha)$.

(3) If X is not 2-closed, then the nilpotency class of T is greater than 2.

Proof. (1) It is enough to observe $Y = TO^2(Y)$.

(2) Let $L/M = O_2(\langle X, Y\rangle/M)$ and $K/N = O_2(L/N)$. Then $O_2(X) \leqq L$ by (1), and 4.1.2 shows that L/N is a $\{2, q\}$-group, where $\pi(Y) = \{2, q\}$, with $O_{2,q}(L/N) = KM/N = K/N \times M/N$. Hence, $C_{O_2(X)}(M/N) = C_{O_2(X)}(KM/K) = O_2(X) \cap KM = O_2(X) \cap K$. Now as $O^2(Y) \nleqq K$, $T \cap K \lhd Y$ by 3.1.1. Hence, $O^2(X) \nleqq K$ by (ii) and so $T \cap K \lhd X$ also. Thus $O_2(X) \cap K \leqq T \cap K \leqq O_2(\alpha)$. Conversely the definitions of L and K show $O_2(\alpha) \leqq O_2(X) \cap K$. Therefore, $O_2(X) \cap K = O_2(\alpha)$.

(3) Set $C = C_X(O_2(X)/O_2(\alpha))$. Then $O^2(X) \nleqq C$ by (ii) and so $T \cap C \leqq O_2(X)$ by 3.1.1. If T has nilpotency class 1 or 2, then $T \leqq C$ by 4.1.4 and so X is 2-closed.

4.3. Let $\alpha = X_T Y \in \Omega$ and assume that the following four conditions are satisfied:

(i) $O_2(\alpha) \neq 1$.

(ii) X is not 2-embedded into Y.

(iii) $O_2(X) \cap O_2(Y) \neq O_2(\alpha)$.

(iv) $T = O_2(X)\, O_2(Y)$.

Set $M = \langle O^2(Y)^{\langle X, Y\rangle}\rangle$ and let M/N be a chief factor of $\langle X, Y\rangle$. Let $P \in \mathrm{Syl}_p(X)$ and $Q \in \mathrm{Syl}_q(Y)$, where $\pi(X) = \{2, p\}$ and $\pi(Y) = \{2, q\}$. Then M/N is an elementary abelian q-group with $m(M/N) \geqq |P/\Phi(P)|\, m(Q/\Phi(Q))$.

Proof. Statements 4.1.2 and 4.2.1 show that we may regard M/N as an irreducible $GF(q)X$-module with respect to the action induced by conjugation. Set $V = O^2(Y)N/N$. Then T leaves V invariant, and Section 3.2 together with (iv) shows that V is an irreducible $GF(q)\, O_2(X)$-submodule with $\dim(V) = m(Q/\Phi(Q))$ and $C_{O_2(X)}(V) = O_2(X) \cap O_2(Y)$. Since $(M/N)_{O_2(X)}$ is completely reducible by Maschke's theorem, it is the direct sum of its homogeneous components, which are transitively permuted by X. Let B be the homogeneous component containing V and let X_B be the stabilizer of B in X. Then $T \leqq X_B$ and there are precisely $|X : X_B|$ homogeneous components. If there is only one homogeneous component, then $C_{O_2(X)}(V) = C_{O_2(X)}(M/N) = O_2(\alpha)$ by 4.2.2, and (iii) is violated. Hence $|X : X_B| > 1$ and thus $X_B \leqq \Phi(P)T$ by 3.2.2. Therefore, $m(M/N) = \dim(M/N) \geqq |X : X_B| \dim(V) \geqq |P/\Phi(P)|\, m(Q/\Phi(Q))$.

DEFINITION. If $\alpha = X_T Y \in \Omega$, we define

$$T_\alpha = [O_2(X), O^2(X)](O_2(X) \cap O_2(Y))[O_2(Y), O^2(Y)],$$

$$X_\alpha = T_\alpha O^2(X), \text{ and } Y_\alpha = T_\alpha O^2(Y).$$

4.4. If $\alpha = X_T Y \in \Omega$, then the following hold:

(1) $T_\alpha \lhd T$ and $X_\alpha \lhd X$.

(2) $T_\alpha \in \mathrm{Syl}_2(X_\alpha)$.

(3) $C_{X_\alpha}(O_2(X_\alpha)) \leqq O_2(X_\alpha)$.

(4) $[O_2(X), O^2(X)] \leqq O_2(X_\alpha)$.

(5) $O_2(X) \cap O_2(Y) = O_2(X_\alpha) \cap O_2(Y_\alpha)$.

(6) $T_\alpha = O_2(X_\alpha)\, O_2(Y_\alpha)$.

(7) $X \leqq_2 Y$ if and only if $T_\alpha \lhd Y$.

Proof. (1) Clearly $T_\alpha \lhd T$. Hence, $X_\alpha = T_\alpha O^2(X) \lhd TO^2(X) = X$.

(2) Statement 3.2.3 shows $T \cap O^2(X) = [O_2(X), O^2(X)]$. Hence, $T \cap X_\alpha = T \cap T_\alpha O^2(X) = T_\alpha(T \cap O^2(X)) = T_\alpha$ and so $T_\alpha \in \mathrm{Syl}_2(X_\alpha)$ by (1).

(3) Part (1) shows $F^*(X_\alpha) \leqq F^*(X) = O_2(X)$ and so $F^*(X_\alpha) = O_2(X_\alpha)$.

(4) As $X_\alpha \geqq T_\alpha \geqq [O_2(X), O^2(X)] \lhd X$, $[O_2(X), O^2(X)] \leqq O_2(X_\alpha)$.

(5) We have $O_2(X) \cap O_2(Y) = O_2(X) \cap O_2(Y) \cap X_\alpha \cap Y_\alpha = O_2(X_\alpha) \cap O_2(Y_\alpha)$ by (1).

(6) This follows from (2), (4), and (5).

(7) If $X \leqq_2 Y$, then $T_\alpha \leqq O_2(Y) \cap Y_\alpha$ and so $T_\alpha = O_2(Y_\alpha)$ by (2). Conversely, if $T_\alpha \lhd Y$, then $[O_2(X), O^2(X)] \leqq T_\alpha \leqq O_2(Y)$.

DEFINITION. If two elements $\alpha = X_T Y$ and $\alpha' = X'_{T'} Y'$ of Ω satisfy the conditions $T' = T_\alpha$, $X = \langle T, X' \rangle$, and $Y = \langle T, Y' \rangle$, then we shall say that α' is smaller than α and write $\alpha' < \alpha$ (primes do not indicate commutator subgroups).

4.5. If $\alpha = X_T Y \in \Omega$, then the following hold:

(1) There are T_α-irreducible subgroups X' and Y' of X_α and Y_α, respectively, such that $X = \langle T, X' \rangle$ and $Y = \langle T, Y' \rangle$.

(2) If X' and Y' are as in (1), then (X', T_α, Y') is an element of Ω smaller than α.

Proof. (1) As $X = TX_\alpha$ and $T_\alpha < X_\alpha$, X is generated by T and the T_α-irreducible subgroups of X_α. Hence there is a T_α-irreducible subgroup X' of X_α which is not contained in the unique maximal subgroup of X containing T and hence $X = \langle T, X' \rangle$. The same argument applies to Y.

(2) It is enough to observe that X' and Y' are contained in $\Gamma(T_\alpha)$ by 4.4.2 and 4.4.3.

4.6. Let $\alpha = X_T Y$ and $\alpha' = X'_{T'} Y'$ be elements of Ω. If $\alpha' < \alpha$, then the following hold:

(1) $\alpha' = \alpha$ if (and only if) $T' = T$.

(2) $X' \leqq X_\alpha$.

(3) $[O_2(X'), O^2(X')] \leqq [O_2(X), O^2(X)] \leqq O_2(X_\alpha) \leqq O_2(X')$.

(4) $T' = O_2(X')\, O_2(Y')$.

(5) $\bigcap O_2(X')^T = O_2(X_\alpha)$.

(6) If X is thin, then $X' = X_\alpha$ and $[O_2(X'), O^2(X')] = [O_2(X), O^2(X)]$.

Proof. (1) If $T' = T$, then $X = \langle T, X' \rangle = X'$ and similarly $Y = Y'$.

(2) As $T' = T_\alpha$ and $O^2(X') \leqq O^2(X)$, $X' = T'O^2(X') \leqq X_\alpha$.

(3) By 3.2.3, $[O_2(X'), O^2(X')] = T' \cap O^2(X') \leqq T \cap O^2(X) = [O_2(X), O^2(X)]$. We have shown $[O_2(X), O^2(X)] \leqq O_2(X_\alpha)$ in Section 4.4. 4.4.2 shows $O_2(X_\alpha) \leqq T_\alpha \leqq X'$, so $O_2(X_\alpha) \leqq O_2(X')$ by (2).

(4) This follows from (3) and 4.4.6, because $O_2(Y_\alpha) \leqq O_2(Y')$ by the symmetry between X and Y.

(5) Set $M = \bigcap O_2(X')^T$. Then $[M, O^2(X')] \leqq [O_2(X'), O^2(X')] \leqq [O_2(X), O^2(X)] \leqq O_2(X_\alpha) \leqq M$ by (3) and 4.4.1. Hence $M \lhd \langle T, O^2(X') \rangle = X$, and therefore $M = O_2(X_\alpha)$.

(6) Assume that X is thin and let $P \in \mathrm{Syl}_p(X)$, where $\pi(X) = \{2, p\}$. Then P is cyclic, and $X' = (P \cap X')T_\alpha$ by (2) and 4.4.2. If $X' \neq X_\alpha$, then $P \cap X' < P$ and so $X' \leqq \Phi(P)T$. But then $\langle T, X' \rangle \neq X$ by 3.2.2, a contradiction. Therefore, $X' = X_\alpha$. Hence, $O^2(X') = O^2(X)$ and $[O_2(X'), O^2(X')] = T_\alpha \cap O^2(X') = T \cap X_\alpha \cap O^2(X') = T \cap O^2(X) = [O_2(X), O^2(X)]$ by 3.2.3.

4.7. Let α be an element of Ω such that $O_2(\alpha) = 1$. If $f(G) \leqq 2$, then there is an element α' of Ω such that $\alpha' < \alpha$ and $O_2(\alpha') = 1$.

Proof. Let $\alpha = X_T Y$ and define Γ_X (resp. Γ_Y) to be the set of all T_α-irreducible subgroups Z of X_α (resp. Y_α) such that $X = \langle T, Z \rangle$ (resp. $Y = \langle T, Z \rangle$). Then Section 4.5 shows that $\Gamma_X \neq \varnothing \neq \Gamma_Y$ and that whenever $X' \in \Gamma_X$ and $Y' \in \Gamma_Y$, (X', T_α, Y') is an element of Ω smaller than α. Set $\Gamma = \Gamma_X \cup \Gamma_Y$ and suppose $O_2(X_1) \cap O_2(X_2) = O_{T_\alpha}(\langle X_1, X_2 \rangle)$ for each pair X_1, X_2 of elements of Γ. Then the intersection D of $O_2(Z)$, Z ranging over Γ, is normal in $\langle T, Z; Z \in \Gamma \rangle = \langle X, Y \rangle$. This is a contradiction because $1 \neq Z(T_\alpha) \leqq D$ by 4.4.3 and 4.6.3. Hence, there are elements X', Y' of Γ with $O_2(X') \cap O_2(Y') \neq O_{T_\alpha}(\langle X', Y' \rangle)$. The X' and Y' must not be 2-embedded into one another, while 4.6.3 shows that two elements of Γ_X or of Γ_Y are 2-embedded into one another. Thus by the symmetry between X and Y, we may assume $X' \in \Gamma_X$, $Y' \in \Gamma_Y$, and that X' is not 2-embedded into Y'. Set $\alpha' = (X', T_\alpha, Y')$. Then $\alpha' \in \Omega$, $\alpha' < \alpha$, and $T_\alpha = O_2(X')\, O_2(Y')$ by 4.6.4. If $O_2(\alpha') \neq 1$, then Section 4.3 gives $n_q(\langle X', Y' \rangle) \geqq 3$, where $\pi(Y') = \{2, q\}$. However, this contradicts our assumption that $f(G) \leqq 2$. Therefore, $O_2(\alpha') = 1$.

DEFINITION. An element $\alpha = X_T Y$ of Ω is said to be thin if both X and Y are thin.

4.8. Let α and α' be elements of Ω with $\alpha' < \alpha$. If α' is thin and $O_2(\alpha') = 1$, then $\alpha = \alpha'$.

Proof. Let $\alpha = X_T Y$ and $\alpha' = X'_{T'} Y'$. The structure of α' is determined by Theorem D (the reader may consult [9, II] for details). Consequently neither X' nor Y' is 2-closed. Hence, neither X nor Y is 2-closed, and $O_2(\alpha) = 1$ by 4.1.3 and 4.4.7. Therefore it suffices to prove that α is thin, for if α is thin, then $T' = T_\alpha = T$ by a remark following Theorem E, whence $\alpha = \alpha'$ by 4.6.1.

Without loss we may assume that it is not $Y'_{T'} X'$ but α' which is a $GL_3(2)$-amalgam, an $Sp_4(2)$-amalgam, ..., or a ${}^2F_4(2)$-amalgam. First, consider the case that α' is a $GL_3(2)$-amalgam or an $Sp_4(2)$-amalgam. Then $O_2(X') \cong O_2(Y') \cong E_4$ or E_8 by definition and so both $[O_2(X), O^2(X)]$ and $[O_2(Y), O^2(Y)]$ have order at most 8 by 4.6.3. Therefore both X and Y are thin by Section 3.6.

Second, consider the case that α' is a ${}^2F_4(2)'$-amalgam or a ${}^2F_4(2)$-amalgam. Then $|[O_2(X'), O^2(X')]^2| = 2^8$ and $|O_2(X')| \leqq 2^{11}$ by [9, II], so $|O_2(X_\alpha)/O_2(X_\alpha)^2| \leqq 8$ by 4.6.3. Hence, $X_\alpha/O_2(X_\alpha)$ is embedded into $GL_3(2)$ by 4.4.3 and consequently X is thin. Similarly we have $|O_2(Y_\alpha)/O_2(Y_\alpha)^2| \leqq 2^5$ and so $Y_\alpha/O_2(Y_\alpha)$ is isomorphic to a $\{2, 5\}$-subgroup of $GL_5(2)$. Thus Y is thin also.

Therefore assume that α' is a $G_2(2)'$-amalgam, a $G_2(2)$-amalgam, an M_{12}-amalgam, or an $\mathrm{Aut}(M_{12})$-amalgam. Set $A = [O_2(X'), O^2(X')]$, $V = A^2$, and $B = [O_2(Y'), O^2(Y')]$. Then

$$A \cong Z_4 \times Z_4$$

by [9, II]. Consequently $[V, O^2(X')] \neq 1$, so $[O_2(X_\alpha)^2, O^2(X)] \neq 1$ by 4.6.3. Also $[O_2(X_\alpha)/O_2(X_\alpha)^2, O^2(X)] \neq 1$ by 4.4.3. Now since $|O_2(X')| \leqq 2^6$ by [9, II], either $O_2(X_\alpha)^2$ or $O_2(X_\alpha)/O_2(X_\alpha)^2$ has order at most 8 by 4.6.3. Therefore X is a thin $\{2, 3\}$-group by Section 3.6. Consequently

$$X' = X_\alpha \qquad \text{and} \qquad A = [O_2(X), O^2(X)]$$

by 4.6.6. Further $|T/O_2(X)| = 2$ by 3.2.1. Now since $B \nleqq O_2(X')$ [8, 3.4], we have $O_2(Y_\alpha) \nleqq O_2(X)$ by 4.6.3. Therefore,

$$T = O_2(X)\, O_2(Y_\alpha).$$

Because of 4.4.5, this gives

$$O_2(Y) = O_2(Y_\alpha)$$

and so by 4.6.3 we have the following sequence:

$$B \leqq [O_2(Y), O^2(Y)] \leqq O_2(Y) \leqq O_2(Y').$$

From now on, we assume that Y is not thin and seek for a contradiction. From [9, II] we know that $|O_2(Y')| \leqq 2^6$ and $|B^2| = 2$. Hence, $|O_2(Y)/O_2(Y)^2| \leqq 2^5$ and, by Section 3.6, $O_2(Y)/O_2(Y)^2$ involves precisely one noncentral chief factor, say W, of Y, which necessarily satisfies $C_Y(W) = O_2(Y)$. Further $2^4 \leqq |W| \leqq 2^5$, but as $|GL_5(2):GL_4(2)|$ is not divisible by 3, $|W| = 2^4$ and $Y/O_2(Y)$ is embedded into $GL_4(2)$. Since Y is not thin, the 3-local structure of $GL_4(2)$ shows that $O^2(Y/O_2(Y)) \cong E_9$ and that $Y/O_2(Y)$ is embedded into a Sylow 3-normalizer of $GL_4(2)$, which is isomorphic to $S_3 \operatorname{wr} Z_2$. Consequently $T/O_2(Y)$ is embedded into D_8, but since Y is not thin, $T/O_2(Y)$ is not elementary by Section 3.5. As $O_2(Y) \leqq O_2(Y') < T' = T_\alpha \lhd T$, we conclude that $T' \geqq T^2 \nleqq O_2(Y)$. Set $K = O_2(X)^2$. Then

$$V \leqq K \leqq T' \cap O_2(X) = O_2(X_\alpha) = O_2(X'),$$

but since $T = O_2(X)\, O_2(Y)$, $K \nleqq O_2(Y)$. Now since $\alpha \neq \alpha'$, $T' < T$ by 4.6.1. Also $|T':O_2(Y')| = 2$. Thus $|O_2(Y'):O_2(Y)| \leqq 2$. Hence, if $K \leqq O_2(Y')$, then $O_2(Y) < KO_2(Y) = O_2(Y') \lhd T$, which contradicts 4.6.5. Therefore

$$K \nleqq O_2(Y').$$

Now X' acts irreducibly on A/V and $A = [O_2(X), O^2(X)] \nleqq K$ as $C_X(O_2(X)) \leqq O_2(X)$. This forces $A \cap K = V$ and so if we set $L/V = C_{O_2(X')/V}(O^2(X'))$, then $K \leqq L$. Thus $L \nleqq O_2(Y')$. From [9, II] we know that this occurs only when α' is an $\operatorname{Aut}(M_{12})$-amalgam. Consequently $|O_2(Y'):B| = 2$ and $|O_2(Y'):O_2(Y')^2| = 8$, while $|O_2(Y):O_2(Y)^2| \geqq |W| = 16$. Therefore $O_2(Y) = B$ and hence $T/B \cong D_8$. Now $A = [O_2(X), O^2(X)] \lhd T$, and we know $|AB/B| = 2$ from [9, II]. Therefore $AB/B = Z(T/B)$. Also $L \lhd T$ as $X' = X_\alpha$, and we know $|LB/B| = 2$ from [9, II]. Therefore $LB/B = Z(T/B)$ also, and we conclude that $LB = AB$. However, we know from [9, II] that this is not the case. This contradiction completes the proof.

4.9. If $X \in \Gamma(T)$, T a 2-subgroup, $O_T(\langle X, N_G(T)\rangle) = 1$, and $|N_G(O_2(X)):X|$ is odd, then a Sylow 2-subgroup of G is dihedral or semidihedral or isomorphic to a Sylow 2-subgroup of $\operatorname{Aut}(Sp_4(2))$.

Proof. Since $O_T(\langle X, N_G(T)\rangle)=1$, no nonidentity characteristic subgroup of T is normal in X, so Theorem F shows that $X=TY$ where $Y=J(X)$ and Y is the direct product of n copies of S_4, m copies of D_8, and l copies of Z_2, $n\geqq 1$, $m\geqq 0$, $l\geqq 0$. Consequently, if we set $U=T\cap Y$, then U is the direct product of $n+m$ copies of D_8 and l copies of Z_2. Hence, $J(T)=U$. Further, Y is generated by U-irreducible subgroups Y_i, $i=1, 2, ..., n$, each of which is the direct product of one copy of S_4, $(n-1)+m$ copies of D_8, and l copies of Z_2, and satisfy the condition $\bigcap_i O_2(Y_i)=O_2(Y)$.

Suppose $O_U(\langle Y_i, Y_j^g\rangle)\neq 1$ for each i, j, and $g\in N_G(T)$. Then as $(Y_i, U, Y_j^g)\in\Omega$ and U has nilpotency class 2, Section 4.2 shows that Y_i and Y_j^g are 2-embedded into one another and so $O_2(Y_i)\cap O_2(Y_j^g)\lhd Y_i$. Hence, $O_2(Y)\cap O_2(Y_j^g)\lhd Y_i$ for each i, j, and $g\in N_G(T)$, because $O_2(Y)\lhd Y_i$. Since $\bigcap_j O_2(Y_j^g)=O_2(Y^g)$, it follows that $O_2(Y)\cap O_2(Y^g)\lhd Y_i$ for each i and $g\in N_G(T)$, and thus $O_2(Y)\cap O_2(Y^g)\lhd Y$ for each $g\in N_G(T)$. But this gives $\bigcap O_2(Y)^{N_G(T)}\lhd\langle Y, N_G(T)\rangle=\langle X, N_G(T)\rangle$, which is a contradiction because $1\neq Z(U)\leqq\bigcap O_2(Y)^{N_G(T)}$.

We have shown that $O_U(\langle Y_i, Y_j^g\rangle)=1$ for some i, j, and $g\in N_G(T)$. We conclude by Theorem D that (Y_i, U, Y_j^g) is a $GL_3(2)$-amalgam or an $Sp_4(2)$-amalgam, and hence $Y_i\cong Y_j^g\cong S_4$ or $S_4\times Z_2$ (see [9, II]). This shows $n=1$, $m=0$, and $l\leqq 1$. Thus $Y\cong S_4$ or $S_4\times Z_2$ and $O_U(\langle Y, Y^g\rangle)=1$. Set $V=O_T(\langle X, X^g\rangle)$ and assume $V\neq 1$. Then $1\neq V\cap\Omega_1(Z(T))\leqq V\cap U$ and, as $V\cap U=V\cap Y=V\cap Y^g$, $V\cap U\lhd\langle Y, Y^g\rangle$. This contradiction shows that $O_T(\langle X, X^g\rangle)=1$. We conclude again by Theorem D that $X\cong S_4$ or $S_4\times Z_2$ and accordingly $T\cong D_8$ or $D_8\times Z_2$. Hence, $\{O_2(X), O_2(X^g)\}$ is the set of the elementary abelian maximal subgroups of T, and if $X\cong S_4\times Z_2$, then $\{Z(X), Z(X^g)\}$ is the set of the maximal subgroups of $Z(T)$ distinct from $[T, T]$. This gives $O_2(X)^{g^2}=O_2(X)$ and $Z(X)^{g^2}=Z(X)$, so there is a 2-element $t\in N_G(T)$ satisfying $O_2(X)^t=O_2(X)^g$ and $Z(X)^t=Z(X)^g$.

Let $\langle t, T\rangle\leqq S\in\mathrm{Syl}_2(G)$. Then as $|N_G(O_2(X)):X|$ is odd, $N_S(O_2(X))=T$ and hence, $C_S(O_2(X))=C_T(O_2(X))=O_2(X)$. If $X\cong S_4$, then $O_2(X)\cong E_4$ and Suzuki's lemma shows that S is either dihedral or semidihedral. Therefore assume $X\cong S_4\times Z_2$ and set $R=N_S(T)$. Then R acts by conjugation on the set $\{O_2(X), O_2(X)^t\}$ of the elementary abelian maximal subgroups of T and the stabilizer of $O_2(X)$ is T. Hence, $|R:T|=2$ and $R=\langle t, T\rangle$. As t moves $O_2(X)$, $R/Z(T)\cong D_8$ and we may assume $t^2\in Z(T)$. Then as t moves $Z(X)$, $\langle t, Z(T)\rangle\cong D_8$ and we may assume $t^2=1$. Thus $C_T(t)\cong Z_4$ and involutions in $R-T$ are all conjugate to t in R. This yields that elementary abelian subgroups of R outside T are of order at most 4. Hence, $J(R)=T$ and so $N_S(R)\leqq N_S(T)=R$. We conclude that $S=R=\langle t, T\rangle$. Let $1\neq a\in Z(X)$ and $b\in O_2(X)-Z(T)$. Then $T=\langle a, a^t, b, b^t\rangle$ and as $1\neq[b, b^t]\in[T, T]$, we have $[b, b^t]=aa^t$. Thus $|S|=2^5$ and S is a homomorphic image of a group Σ presented on generators α, β, τ with defining relations $\alpha^2=\beta^2=\tau^2=1$, $\alpha\beta=\beta\alpha$, $\alpha\beta^\tau=\beta^\tau\alpha$, $\alpha\alpha^\tau=\alpha^\tau\alpha$, $[\beta, \beta^\tau]=\alpha\alpha^\tau$. To complete the proof it is enough to observe that these relations imply $|\Sigma|\leqq 2^5$ and that a Sylow 2-subgroup of $\mathrm{Aut}(Sp_4(2))$ has order 2^5 and has three generators satisfying the above relations.

5. Proofs of Theorems B and C

Let G be a group satisfying the hypothesis of Section 4. In order to prove Theorem B, let $\alpha=(X, S, Y)$ be an extremal S_2-amalgam in G with $S\in \mathrm{Syl}_2(G)$. Then since G is of characteristic 2 type and $S\in\mathrm{Syl}_2(G)$, we have $C_X(O_2(X))\leqq O_2(X)$ and $C_Y(O_2(Y))\leqq O_2(Y)$. Thus $\alpha\in\Omega$ and $O_2(\alpha)=1$. As we are assuming $f(G)\leqq 2$, Section 4.7 shows that there is a sequence $\alpha_0=\alpha, \alpha_1, \alpha_2, ..., \alpha_n, ...$ of elements of Ω such that $\alpha_{n+1}<\alpha_n$ and $O_2(\alpha_n)=1$ for each $n\geqq 0$. Let $\alpha_n=(X_n, T_n, Y_n)$, $n\geqq 0$. Then as $T_{n+1}\leqq T_n$ for each n, there is a nonnegative integer m such that $T_{m+1}=T_m$. Theorem E now shows that α_m is thin, so by Section 4.8, $\alpha=\alpha_m$ and α is thin.

In order to prove Theorem C, let $S\in\mathrm{Syl}_2(G)$ and assume that S is contained in a unique maximal 2-local subgroup, say M, of G. Assume that M is not strongly embedded in G and pick a 2-subgroup T of M so that $|N_M(T)|_2$ is maximal subject to the conditions $T\neq 1$ and $N_G(T)\nleqq M$. Let $R\in\mathrm{Syl}_2(N_M(T))$. Then as $R\notin\mathrm{Syl}_2(M)$, $|R|<|N_M(R)|_2$ and so $N_G(T)\cap N_G(R)\leqq N_M(T)$. Thus $R\in\mathrm{Syl}_2(N_G(T))$. As $N_G(T)\nleqq M$, there is an R-irreducible subgroup X of $N_G(T)$ such that $X\nleqq M$. The X is contained in $\Gamma(R)$ because $R\in\mathrm{Syl}_2(N_G(T))$. Set $V=O_R(\langle X, N_G(R)\rangle)$. Then $|R|<|N_M(R)|_2\leqq|N_M(V)|_2$ and $N_G(V)\nleqq M$, so $V=1$. Further $N_G(O_2(X))\nleqq M$ and $R\leqq N_M(O_2(X))$, so $R\in\mathrm{Syl}_2(N_G(O_2(X)))$. We can now apply Section 4.9 to conclude that (C.2) is satisfied.

References

1. Z. Janko, Nonsolvable finite groups all of whose 2-local subgroups are solvable, *J. Algebra* **21** (1972), 458–517.
2. F. Smith, Finite simple groups all of whose 2-local subgroups are solvable, *J. Algebra* **34** (1975), 481–520.
3. M. J. Collins, Ed., "Finite Simple Groups II," Academic Press, London, 1980.
4. B. Cooperstein and G. Mason, Eds., "The Santa Cruz Conference on Finite Groups," Amer. Math. Soc., Providence, RI, 1980.
5. K. Gomi, A note on the thin finite simple groups, *J. Algebra* **102** (1986), 194–198.
6. J. G. Thompson, Nonsolvable finite groups all of whose local subgroups are solvable, I–VI, *Bull. Amer. Math. Soc.* **74** (1968), 383–437; *Pacific J. Math.* **33** (1970), 431–536; *Pacific J. Math.* **39** (1971), 483–534; *Pacific J. Math.* **48** (1973), 511–592; *Pacific J. Math.* **50** (1974), 215–297; *Pacific J. Math.* **51** (1974), 573–630.

7. K. GOMI, On the 2-local structure of groups of characteristic 2 type, *J. Algebra* **108** (1987), 492–502.
8. K. GOMI, Pairs of groups having a common 2-subgroup of prime indices, *J. Algebra* **97** (1985), 407–437.
9. K. GOMI AND Y. TANAKA, On pairs of groups having a common 2-subgroup of odd indices, I–II, *Sci. Papers College Arts Sci. Univ. Tokyo* **35** (1985), 11–30; *Sci. Papers College Arts Sci. Univ. Tokyo* **40** (1990), 37–47.
10. Y. TANAKA, Certain pairs of finite groups with a common 2-subgroup, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **33** (1986), 247–272.
11. Y. TANAKA, "Amalgams of Quasithin Solvable Groups," Doctoral dissertation, University of Tokyo, Tokyo, 1990.
12. K. GOMI, A pushing up theorem for groups of characteristic 2 type, *Sci. Papers College Arts Sci. Univ. Tokyo* **37** (1987), 73–102.